MATH 4600, Section 1 Worksheet No. 10 January 28, 2008
Mathematics Seminar
Spring 2008

First of all, make a serious attempt to solve the given problem. If after that you feel you need some help, take a look at the hints.

Challenging Problems – Systematical Systems

1. (Source: some sleepless nights.) Both quadratic equations $3x^2 + 4x + 1 = 0$ and $4x^2 + 5x + 1 = 0$ have rational roots. Can you generalize? Can you prove your generalized guess? Can you generalize your generalization (to, say, rational numbers)?

2. (Source: Ignat'ev, *In the realm of sharpness.*) A given positive integer ends in 2. If we shift this digit to the first place, the number is doubled. Find this number.

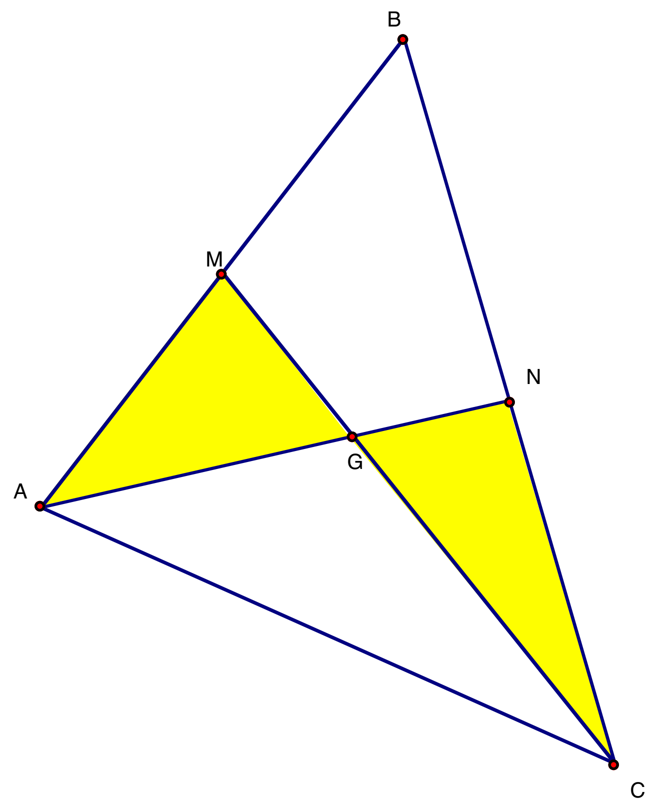
3. (Source: Ignat'ev, again.) Three peasants were walking together. They stopped at a coach inn to rest and have lunch. They ordered the landlady to bake some potatos and went to sleep. She baked the potatos and left the bowl on the table. The first peasant woke up and ate his share. When the second peasant woke up, he didn't know that one of his companions had already eaten his share, so he ate a third of the remaining potatos and went to sleep again. When the third peasant woke up, he thought he had been the first to awake, and ate a third of the remaining potatoes. By now his other two companions woke up and saw that there remained 8 potatoes, and all realized what had happend. How many potatoes were initially given, how many did each one eat, and how many should each eat to get a fair share?

4. (Source: Kostrikin, Introduction to algebra. Some linear algebra knowledge required.) The numbers $1798 = 31 \cdot 58$, $2139 = 31 \cdot 69$, $3255 = 31 \cdot 105$, and $4867 = 31 \cdot 157$ are all diviible by 31. Without doing lengthy computations, show that the determinant

$$\begin{vmatrix} 1 & 7 & 9 & 8 \\ 2 & 1 & 3 & 9 \\ 3 & 2 & 5 & 5 \\ 4 & 8 & 6 & 7 \end{vmatrix}$$

is also divisible by 31.

5. (Source: my deranged mind. I am sure this is well known, from basic geometry.) In a generic triangle ABC let G be the point of intersection of the medians CN and AM (so that M is the midpoint of AB and N is the midpoint of BC; see figure below). Prove that the areas of the triangles AMG and GNC are equal. What is the ratio of each of these areas to the total area of ABC?



6. (Source: Perel'man, *Funny Problems.*) The lights went out in the dorm. Tom lighted two candles and continued to work using them until electricity was restored.

On the following day Tom wanted to know how long the blackout had lasted. He didn't notice the time when the lights went off and when they came back again. He didn't recall the lengths of the remaining pieces of the candles. He only remembered that one was thick and could burn for 5 hours, and the other was thinner and would take for 4 hours to burn completely. The problem is, whe couldn't find the pieces of the candles—his roommate Bob had thrown them away. "How big were the pieces?" Tom asked his friend. Bob replied "I don't know, but I recall that one piece was four times bigger than the other.

How long had the candles burned?

7. (Source: Perel'man, again.) Uncle Tom came for a visit. Little Ben came to say hello, together with his sister Anna. The little boy produly announced his uncle that he was twice as old as his sister.
Next came Becky and her father, who told Tom that the two girls together were twice as old as the boy.
When Charles came from school the father announced that the two boys together are twice as old as the two girls together.
Last came Daisy. When he met Tom, she happily exclaimed "Uncle, you came on my birthday! I am 21 years old today!"
"You know what" added the fater, "I have just figured out that my three daughters together are twice as old as my two sons."
How old was each child?

8. (Source unknown.) Two people are taking. The older one says "I am twice as old as you were when I was your age. When you are my age, our ages combined will be 90 years."
How old is each person?

9. (Source: still Perel'man.) Two pieces were found of a chain consisting of identical links. The metal making up the links was $^1/_2$cm thick. When one piece was extended, it was 36cm long, and the other was 22cm long. The long piece had 6 links more than the short one.
How many links were in each piece?

10. (Source: Martin Gardner, *Entertaining Mathematical Puzzles.*) A straight line is *self-congruent*, in the sense that any of its segments can be exactly fitted to any other segment of the same length. The same is true for the circumference of any given circle. Can you find a third class of self-congruent lines?

11. (Source: Martin Gardner again.) Put a sweater on (if you don't have one already). Tie your wrists together with a (reasonably long) piece of rope. Can you take off your sweater, turn it inside out, and put it back on again?

Hints

1. First try $nx^2 + (n+1)x + 1 = 0$. What are the assumptions on n? Does it have to be an integer? And what if you try $(n+1)x^2 + nx^2 + \text{something} = 0$?
2. What should the second to last digit be? And what about the preceding digit?
3. Solve the problem backwards.
4. Use suitable properties of determinants.
5. Find the ratio of the area of AMG to that of ABC.
6. It helps to draw the two candles side by side, and figuer out how they burn.
7. Maybe some algebra will help.
8. Read the problem carefully. The statement is a little tricky. No guesswork, please.
9. To solve this problem it helps a lot to draw several links.
10. Think outside the box.
11. It can be done. No cutting the rope or the sweater, no un-knotting, no trying to squeeze the sweater between the rope and the wrist, there is no slack there.