MATH 4600, Section 1 | Worksheet No. 10 | January 28, 2008
Mathematics Seminar
Spring 2008

First of all, make a serious attempt to solve the given problem. If after that you feel you need some help, take a look at the hints.

Challenging Problems – Systematical Systems

1. (Source: some sleepless nights.) Both quadratic equations $3x^2 + 4x + 1 = 0$ and $4x^2 + 5x + 1 = 0$ have rational roots. Can you generalize? Can you prove your generalized guess? Can you generalize your generalization (to, say, rational numbers)?

2. (Source: Ignat'ev, *In the realm of sharpness.*) A given positive integer ends in 2. If we shift this digit to the first place, the number is doubled. Find this number.

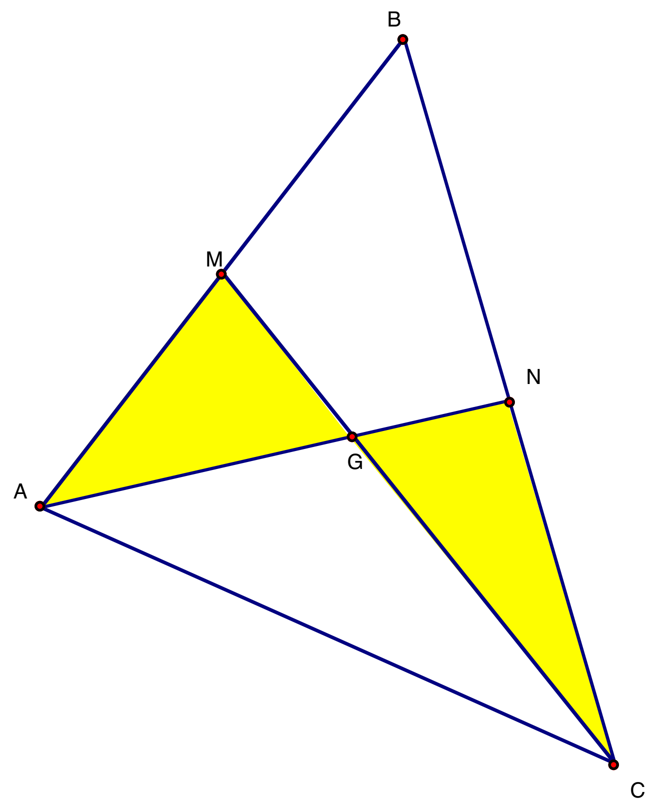
3. (Source: Ignat'ev, again.) Three peasants were walking together. They stopped at a coach inn to rest and have lunch. They ordered the landlady to bake some potatos and went to sleep. She baked the potatos and left the bowl on the table. The first peasant woke up and ate his share. When the second peasant woke up, he didn't know that one of his companions had already eaten his share, so he ate a third of the remaining potatos and went to sleep again. When the third peasant woke up, he thought he had been the first to awake, and ate a third of the remaining potatoes. By now his other two companions woke up and saw that there remained 8 potatoes, and all realized what had happend. How many potatoes were initially given, how many did each one eat, and how many should each eat to get a fair share?

4. (Source: Kostrikin, Introduction to algebra. Some linear algebra knowledge required.) The numbers $1798 = 31 \cdot 58$, $2139 = 31 \cdot 69$, $3255 = 31 \cdot 105$, and $4867 = 31 \cdot 157$ are all diviible by 31. Without doing lengthy computations, show that the determinant

$$\begin{vmatrix} 1 & 7 & 9 & 8 \\ 2 & 1 & 3 & 9 \\ 3 & 2 & 5 & 5 \\ 4 & 8 & 6 & 7 \end{vmatrix}$$

is also divisible by 31.

5. (Source: my deranged mind. I am sure this is well known, from basic geometry.) In a generic triangle ABC let G be the point of intersection of the medians CN and AM (so that M is the midpoint of AB and N is the midpoint of BC; see figure below). Prove that the areas of the triangles AMG and GNC are equal. What is the ratio of each of these areas to the total area of ABC?



6. (Source: Perel'man, *Funny Problems*.) The lights went out in the dorm. Tom lighted two candles and continued to work using them until electricity was restored.

On the following day Tom wanted to know how long the blackout had lasted. He didn't notice the time when the lights went off and when they came back again. He didn't recall the lengths of the remaining pieces of the candles. He only remembered that one was thick and could burn for 5 hours, and the other was thinner and would take for 4 hours to burn completely. The problem is, whe couldn't find the pieces of the candles—his roommate Bob had thrown them away. "How big were the pieces?" Tom asked his friend. Bob replied "I don't know, but I recall that one piece was four times bigger than the other.

How long had the candles burned?

7. (Source: Perel'man, again.) Uncle Tom came for a visit. Little Ben came to say hello, together with his sister Anna. The little boy produly announced his uncle that he was twice as old as his sister.
Next came Becky and her father, who told Tom that the two girls together were twice as old as the boy.
When Charles came from school the father announced that the two boys together are twice as old as the two girls together.
Last came Daisy. When he met Tom, she happily exclaimed "Uncle, you came on my birthday! I am 21 years old today!"
"You know what" added the fater, "I have just figured out that my three daughters together are twice as old as my two sons."
How old was each child?

8. (Source unknown.) Two people are taking. The older one says "I am twice as old as you were when I was your age. When you are my age, our ages combined will be 90 years."
How old is each person?

9. (Source: still Perel'man.) Two pieces were found of a chain consisting of identical links. The metal making up the links was $^1/_2$cm thick. When one piece was extended, it was 36cm long, and the other was 22cm long. The long piece had 6 links more than the short one.
How many links were in each piece?

10. (Source: Martin Gardner, *Entertaining Mathematical Puzzles.*) A straight line is *self-congruent*, in the sense that any of its segments can be exactly fitted to any other segment of the same length. The same is true for the circumference of any given circle. Can you find a third class of self-congruent lines?

11. (Source: Martin Gardner again.) Put a sweater on (if you don't have one already). Tie your wrists together with a (reasonably long) piece of rope. Can you take off your sweater, turn it inside out, and put it back on again?

Hints

1. First try $nx^2 + (n+1)x + 1 = 0$. What are the assumptions on n? Does it have to be an integer? And what if you try $(n+1)x^2 + nx^2 + \text{something} = 0$?
2. What should the second to last digit be? And what about the preceding digit?
3. Solve the problem backwards.
4. Use suitable properties of determinants.
5. Find the ratio of the area of AMG to that of ABC.
6. It helps to draw the two candles side by side, and figuer out how they burn.
7. Maybe some algebra will help.
8. Read the problem carefully. The statement is a little tricky. No guesswork, please.
9. To solve this problem it helps a lot to draw several links.
10. Think outside the box.
11. It can be done. No cutting the rope or the sweater, no un-knotting, no trying to squeeze the sweater between the rope and the wrist, there is no slack there.

Answers

1. The discriminant of the corresponding equation is a perfect square, both when n is an integer and when n is a rational number. Setting $n = {}^p/_q$ leads to interesting families of quadratic equations with rational roots. For the second family, try "something" $= -1$.

2. The second to last digit should be $2 \cdot 2 = 4$. The previous one should be 8, the previous one, 6 (since $8 \cdot 8 = 16$), the previous one should be 3 (why?), the previous one, 7, and so on. The number should start with 1. Therefore we stop when after doubling the previous digit and adding 1 to the difits of the previous member of the sequence, we get 1. The first solution is

$$105\,263\,157\,894\,736\,842.$$

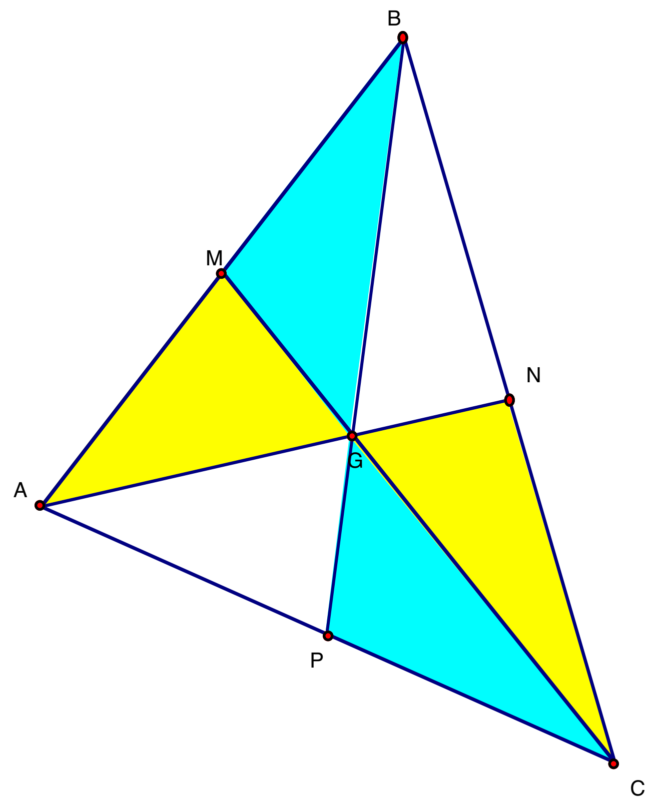
Continuing the process, one gets infinitely many solutions. It is not hard to see that each solution wil consist of several repetitions of combinations of digits we have already found.

3. The third peasant left 8 potatoes; therefore, he ate 12. The second peasant left 12 potatoes, so he ate 6. Hence the first one left 18 potatoes and ate 9.

There were 27 potatoes given initially. The first peasant already ate his share of 9 potatoes. Out of the 8 remaining potatoes, 3 should be given to the first peasant and 5 to the third one.

4. Multiply the first column by 1000, the second by 100, the third by 10, and add them all to the fourth. Then in the fourth column we get the numbers 1798, and so on. The new determinant should be equal to the old one. This is a consequence of several properties, namely: (a) if a determinant has two equal columns, it is equal to zero; (b) if we multiply all the elements of a column by a given number, the determinant gets mutliplied by that number; (c) if a column is equal to the sum of two columns, the determinant is equal to the sum of two determinants, each having one of the (added) columns in place of the composite column. Now, using again one of the listed properties, take out the common factor 31 from the fourth column.

5. Since AM is half of AB and MG is a third of MC then the base AM of AMG is half of the base AB of ABC, and the height corresponding to that base is a third of the height of ABC corresponding to AB. Therefore, the area of AMG is equal to ${}^1/_6$ of the total area of ABC. A similar argument proves that the area of GNC is also ${}^1/_6$ of that of ABC. Consequently, the areas of AMG and GNC are equal.



By the way, if we add the third median BP, we get a nice picture (see picture above): ABC is divided into six smaller triangles, each having the same area. This suggests another possible way of proving the result: the area of ANB is equal to that of ANC, and they are both half the area of ABC. Applying a similar argument to the other two sides of ABC and their midpoints, one again gets that all six smaller triangles must have equal area.

6. At any given moment, the length of the part of the thinner candle that has burned out is equal to ${}^5\!/_4$ of the length of the burnt part of the thicker one. In other words, the part of the thinner candle that has burned out is longer by ${}^1\!/_4$ of the burnt part ot the thicker candle.

On the other hand, the length of this excess candle is equal to ${}^3\!/_4$ of the length of the remaining piece of the thick candle. Hence, ${}^4\!/_4$ of he remaining thick piece—that is, the whole piece—makes up ${}^1\!/_4 \times {}^4\!/_3 = {}^1\!/_3$ of the thick candle.

Thus, the remaining piece of the thick candle is equal to ${}^1\!/_3$ of the part that burnt, or ${}^1\!/_4$ of the entire candle. We conclude that ${}^3\!/_4$ of the thick candle burnt. Since it takes 5 hours for the whole candle to burn, $^3/_4$ of that took

$$\frac{5\times 3}{4}=\frac{15}{4}=3\,\frac{3}{4}\text{ hours.}$$

7. Let us call A = Anna's age, B = Becky's, C = the age of Charles, D = that of Daisy, and Be, that of Ben. Then the successive conditions yield

$$B=2A,\quad A+B=2Be,\quad Be+C=2(B+A),$$

$$D+A+B=2(Be+C).$$

But then $A+B=2Be=4A$, whence $B=3A$. The next condition then becomes

$$2A+C=2(3A+A),$$

and therefore $C=6A$. Finally, for the last condition we get

$$D+A+3A=2(2A+6A),$$

which implies $D=12A$. Since $D=21$, we get $A={}^{21}/_{12}={}^7/_4$, so Anna was 1 and $^3/_4$ years. Next $B=3A={}^{21}/_4=5^1/_4$. For Ben we get $Be=2A={}^7/_2=3^1/_2$. Finally, $C=6A={}^{21}/_2=10^1/_2$. Thus, Ben was 3 and $^1/_2$ years, Anna was 1 and $^3/_4$; Becky was $5^1/_4$, Charles was $10^1/_2$ and Daisy, of course, was 21.

8. There are several possible ways to solve the problem. Here is one. Let x be the age of the older person, and y, that of the younger one. Call $\Delta=y-x$ the difference in their ages. Then the ages now are (x,y), in the past they were $(y,y-\Delta)$, and in the future they will be $(x+\Delta,x)$.
The first condition then translates into $x=2(y-\Delta)$, and the second one becomes $(x+\Delta)+x=90$, or $2x+\Delta=90$. Finally, by definition $\Delta=x-y$. Solving the system, we get $\Delta=10$, $x=40$, $y=30$.

9. Once you draw several links, you realize that the length of the extended chain consists of the full length of the first link, to which one adds, for each new link, the full length of the link *minus* twice the thickness of the material.
Further, we know that one piece is 14 cm longer than the other. Dividing 14 by 6 we get $2^1/_3$. This is the thickness of one link minus twice its thickness. Since this thickness is $^1/_2$cm, the total length of each link is $2^1/_3+{}^1/_2+{}^1/_2=3^1/_3$cm.
Now we can find how many links there are in each piece. Drawing a picture we see that if from a 36cm-long chain we remove the double thickness of the first link—that is, 1cm—and divide the difference by $2^1/_3$, we get the number of links in this chain: 35 divided by $2^1/_3$ equals 15.

Analogously, for the 21cm-long piece we get 21 divided by $2^1/_3$ equals 9.

10. Another self-congruent line lies no longer in the plane, but in space; it is a *circular helix*, a line that spirals on the surface of a right circular cylinder, going "up" (if the cylinder is vertical) with constant speed. If the cylinder has radius R, a possible parametric equation of a helix is $(R\cos t, R\sin t, at)$, where a is any constant (positive, if we want the spiral to go up as t increases), $-\infty < t < \infty$.

11. Pull the sweater over your head, reversing it as you do so, and allow it to hang on the fope. Reverse the sweater again by pushing it through one of the sleeves. Put it on again, reversing the sweater a third time.